

# ARTICULATED ARM AND SPECIAL MULTI-FLAGS (CORRECTED VERSION)

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**ABSTRACT.** In this paper we give a kinematical illustration of some distributions called special multi-flags distributions. Precisely we define the kinematic model in angular coordinates of an articulated arm constituted of a series of  $(n + 1)$  segments in  $\mathbb{R}^{k+1}$  and construct the special multi-flag distribution associated to this model.

**Keywords:** Goursat distributions, car with  $n$  trailers, special multi-flags distributions, articulated arm.

AMS classification: 53, 58, 70, 93.

## 1. Introduction

The kinematic evolution of a car towing  $n$  trailers can be described by a Goursat distribution on the configuration space  $M = \mathbb{R}^2 \times (\mathbb{S}^1)^{n+1}$ . A Goursat distribution is a rank- $(l - s)$  distribution on a manifold  $M$  of dimension  $l \geq 2 + s$ , such that each element of its flag of Lie squares,

$$D = D^s \subset D^{s-1} = [D^s, D^s] \subset \dots \subset D^{j-1} = [D^j, D^j] \subset \dots \subset D^0 = TM$$

is of codimension 1 in the following one.

Since 2000, Goursat distributions were generalized in many works ([7], [12] [13], [14], [15], [20]). Special  $k$ -flags ( $k \geq 2$ ), which are considered to be extensions of Goursat flags, were defined in [7],[14], and [20] in several equivalent ways. All these approaches can be reduced to one transparent definition (see [1], [24]). A special  $k$ -flag of length  $s$  on a manifold  $M$  of dimension  $(s + 1)k + 1$  is a sequence of distributions

$$D^s \subset D^{s-1} = [D^s, D^s] \subset \dots \subset D^{j-1} = [D^j, D^j] \subset \dots \subset D^0 = TM$$

such that the respective dimensions of  $D^s, D^{s-1}, \dots, D^0$  are  $k + 1, 2k + 1, \dots, (s + 1)k + 1$ , for  $j = 1, \dots, s - 1$ , the Cauchy-characteristic subdistribution  $L(D^j)$  of  $D^j$  is included in  $D^{j+1}$  of constant corank one,  $L(D^s) = 0$ , and there exists a completely integrable subdistribution  $F \subset D^1$  of corank one in  $D^1$ . The integer  $k$  is called width.

The purpose of this work is to show that the problem of modeling car towing  $n$  trailers can be generalized to the problem of modeling kinematic problem for an “articulated arm” constituted of  $(n + 1)$  segments in  $\mathbb{R}^{k+1}$ , such that to this model is naturally associated a special  $k$ -flag.

In the following, an “articulated arm” of length  $(n + 1)$  is a series of  $(n + 1)$  segments  $[M_i, M_{i+1}]$ ,  $i = 0, \dots, n - 1$ , in  $\mathbb{R}^{k+1}$ , keeping a constant length  $l_i$ , and the articulation occurs at points  $M_i$ , for  $i = 1, \dots, n$ .

It is proposed to study the kinematic evolution of such a mechanical system under the constraint that the velocity of each point  $M_i$  is collinear with the segment  $[M_i, M_{i+1}]$ , for  $i = 0, \dots, n$ . Such a system is also studied in [10] and is called a “n-bar system”. In this paper we define precisely the kinematic evolution of this mechanical system in term of hyperspherical coordinates and we construct the special multi-flag naturally associated to this model.

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For  $k = 1$ , an articulated arm of length  $n$  is a modeling problem of a car with  $n$  trailers. (see [3]). When the number of trailers is large, this problem can be considered as an approximation of the “nonholonomic snake” in the plane (see [23] for instance). For  $k > 1$  we can also consider a “snake” in  $\mathbb{R}^{k+1}$  (see [22] for a complete description). Again, an articulated arm of length  $n$ , for  $n$  large, can be considered as a discretization of a nonholonomic snake in  $\mathbb{R}^{k+1}$ . For instance, in  $\mathbb{R}^3$ , some problems of “towed cable” can model in such a way ([17], [23])

In section 2, we recall the classic context of the car with  $n$  trailers and its interpretation in terms of Goursat distribution. The articulated arm system is developed in section 3 and also we show how to associate a special multi-flags to such a system in cartesian coordinates. In section 4, we gives a version of the kinematic evolution of an articulated arm in terms of angular coordinates and we get a generalization of the classical model of the car with  $n$  trailers. The last two sections are devoted to the proofs of the results.

## 2. The car with n trailers

In this section we will recall some fundamental results about the system of the car with  $n$  trailers and its relation with the Goursat distribution. All these results are now classical and can be found in a large number of papers as [2], [3], [11], [18], [26] and many others .

### 2.1. Notations and equations.

A car with  $n$  trailers is a configuration of  $(n+1)$  trailers in the  $\mathbb{R}^2$ -plane, denoted by  $M_0, M_1, \dots, M_n$ , and keeping a constant length between each two trailers. It is proposed to study the kinematic evolution of the trailer  $M_n$  with the constraint that the motion is controlled by the evolution of  $M_0$  which symbolize the car. We will use the same representation as M. Fliess [2] and O.J. Sordalen [26] where the car is represented by two driving wheels connected by an axle. It’s a kinematic problem with non integrable constraints (i.e. a nonholonomic system) due to the rolling without sliding of the wheels. The configuration space of the system is characterized by the two dimensional coordinates of  $M_n$  and  $(n+1)$  angles, whereas there are only two inputs, namely one tangential velocity and one angular velocity which represent the action on the steering wheel and on the accelerator of the car. Consider the system of the car with  $n$  trailers and suppose that the distances  $R_r$  between the different trailers are all equal to 1. We choose as a reference point of a body  $M_{n-r}$  the midpoint  $m_r$  between the wheels; its coordinates are denoted by  $x_r$  and  $y_r$  in a given cartesian frame of the plane;  $\theta_r$  is the angle between the main axis of  $M_{n-r}$  and the  $x$ -axis of the frame. So, the set of all positions of the car with  $n$  trailers is included in a  $3(n+1)$ -dimensional space. This system is submitted to  $2n$  holonomic links which give, in the previous space, the  $2n$  following equations:

$$(1) \quad \begin{aligned} x_r - x_{r-1} &= \cos \theta_{r-1} \\ y_r - y_{r-1} &= \sin \theta_{r-1} \end{aligned}$$

The configuration space of this problem is a submanifold of dimension  $(n+3)$  which is parameterized by  $q = (x_0, y_0, \theta_0, \dots, \theta_n)$  where :

- $(x_0, y_0)$  are the coordinates of the last trailer  $M_n$ .
- $\theta_n$  is the orientation of the car (the trailer  $M_0$ ) with respect to the  $x$ -axis.
- $\theta_r, 0 \leq r \leq n-1$ , is the orientation of the trailer  $(n-r)$  with respect to the  $x$ -axis.

The configuration space can thus be identified to  $\mathbb{R}^2 \times (\mathbb{S}^1)^{n+1}$ .

The velocity parameters are  $\dot{x}_0, \dot{y}_0, \dot{\theta}_0, \dots, \dot{\theta}_n$ . There are only two inputs, namely the “angular velocity”  $w_n$  and the “tangential velocity”  $v_n$  of the midpoint of the guiding wheels associated to the action of the car (see [3]).

Assume that the contacts between the wheels and the ground are pure rolling, it is then submitted to the classical nonholonomic links:

$$(2) \quad \dot{x}_r \sin \theta_r - \dot{y}_r \cos \theta_r = 0$$

There are  $(n + 1)$  kinematic equality constraints, one for each trailer. In order to establish these constraints, we can represent the points  $m_r$ ,  $r = 0, \dots, n$ , in the complex plane, i.e,  $m_r = x_r + iy_r$ . The geometric constraint between two consecutive trailers is written as:

$$m_r = m_{r-1} + e^{i\theta_{r-1}} \quad \text{for } r \neq 0$$

By induction, we have the following equation:

$$(3) \quad m_r = m_0 + \sum_{l=0}^{r-1} e^{i\theta_l}$$

The kinematic constraint of  $M_{n-r}$  is :

$$\dot{m}_r = \lambda_r e^{i\theta_r}$$

which is equivalent to :

$$\mathcal{I}(e^{(-i\theta_r)} \dot{m}_r) = 0$$

where  $\mathcal{I}(z)$  denotes the imaginary part of  $z$ . Combining this characterization with the derivative of (3) and using the linearity of  $\mathcal{I}$ , we obtain the kinematic constraints:

$$(4) \quad -\dot{x}_0 \sin \theta_r + \dot{y}_0 \cos \theta_r + \sum_{j=0}^{r-1} \dot{\theta}_j \cos(\theta_j - \theta_r) = 0 \quad r = 0, \dots, n$$

Combining  $\dot{m}_r = \lambda_r e^{i\theta_r}$  with the derivative of

$$|m_{r+1} - m_r|^2 = 1$$

we obtain

$$\lambda_r = \lambda_{r+1} (\cos \theta_{r+1} - \cos \theta_r)$$

and by induction:

$$\lambda_r = \lambda_n \cos(\theta_n - \theta_{n-1}) \cdots \cos(\theta_{r+1} - \theta_r)$$

so

$$\dot{m}_r = \lambda_n \left( \prod_{j=r+1}^n \cos(\theta_j - \theta_{j-1}) \right) e^{i\theta_r}$$

where  $\lambda_n = v_n$  is the tangential velocity of the car  $M_0$ .

*The evolution of the system of car with  $n$  trailers can be given by the following controlled system with two controls  $v_n$  ("tangential velocity") and  $w_n$  ("normal velocity") of  $M_0$ :*

$$(5) \quad \begin{cases} \dot{x}_0 = v_0 \cos \theta_0 \\ \dot{y}_0 = v_0 \sin \theta_0 \\ \dot{\theta}_0 = v_1 \sin(\theta_1 - \theta_0) \\ \dots \\ \dot{\theta}_r = v_{r+1} \sin(\theta_{r+1} - \theta_r) \\ \dots \\ \dot{\theta}_{n-1} = v_n \sin(\theta_n - \theta_{n-1}) \\ \dot{\theta}_n = w_n \end{cases}$$

The "tangential velocity"  $v_r$  of the body  $M_{n-r}$  is given by :

$$v_r = \prod_{j=r+1}^n \cos(\theta_j - \theta_{j-1}) v_n$$

## 2.2. Goursat flag.

Given a smooth distribution  $D$  on a manifold  $M$  we will use the standard notation  $[D, D]$  to denote the smooth distribution generated by the vector fields tangent to  $D$  and the Lie brackets  $[X, Y]$ , of any pair  $(X, Y)$  of vector fields tangent to  $D$ .

**Definition 2.1.** *A Goursat flag of length  $s$  on a manifold  $M$  of dimension  $l \geq s + 2$  is a sequence of distributions on  $M$*

$$D^s \subset D^{s-1} \subset \dots \subset D^3 \subset D^2 \subset D^1 \subset D^0 = TM \quad s \geq 2 \quad (F)$$

satisfying the following Goursat conditions

$$\begin{aligned} 1) \text{ corang } D^i &= i \quad i = 1, 2, \dots, s \\ 2) D^{i-1} &= [D^i, D^i] \quad i = 1, 2, \dots, s \end{aligned} \quad (G)$$

Each  $D^i(p)$  is a subspace of  $T_p M$  of codimension  $i$ , for any point  $p \in M$ . It follows that  $D^{i+1}(p)$  is a hyperplane in  $D^i(p)$ , for any  $i = 0, 1, \dots, s-1$  and  $p \in M$ .

**Definition 2.2.** *We call any distribution  $D^i$  of corank  $i \geq 2$  in a Goursat flag (F) a Goursat distribution.*

To each flag (F) of Goursat distributions we associate a flag of “Cauchy-characteristic” subdistributions

$$L(D^s) \subset L(D^{s-1}) \subset \dots \subset L(D^3) \subset L(D^2) \subset L(D^1) \quad (L)$$

where  $L(D)$  is the subdistribution of  $D$  generated by the set of vector fields  $X$  tangent to  $D$  such that  $[X, Y]$  is tangent to  $D$  for all  $Y$  tangent to  $D$ .  $L(D)$  is called the Cauchy-characteristic distribution of  $D$ .

**Lemma 2.1.** (Sandwich lemma)[11]: *Let  $D$  be any Goursat distribution of corank  $s \geq 2$  on a manifold  $M$ , and  $p$  be any point of  $M$ . Then*

$$L(D)(p) \subset L([D, D])(p) \subset D(p),$$

with  $\dim L(D)(p) = \dim D(p) - 2$ ,  $\dim L([D, D])(p) = \dim D(p) - 1$ .

It follows a relation between the Goursat flag and its flag of Cauchy-characteristic subdistributions:

$$\begin{array}{ccccccccccc} D^s & \subset & D^{s-1} & \subset & \dots & \subset & D^3 & \subset & D^2 & \subset & D^1 & \subset & D^0 \\ \cup & & \cup & & & & \cup & & \cup & & & & \\ L(D^s) & \subset & L(D^{s-1}) & \subset & L(D^{s-2}) & \subset & \dots & \subset & L(D^2) & \subset & L(D^1) \end{array}$$

Each inclusion here is a codimension one inclusion of subbundles of the tangent bundle.  $L(D^i)$  is an involutive regular distribution on  $M$  of codimension  $i + 2$ .

## 2.3. Goursat flag associated to the car with $n$ trailers.

Let  $f_r^n = \prod_{j=r+1}^n \cos(\theta_j - \theta_{j-1})$ ,

and  $v_r = f_r^n v_n$  for  $r = 1, \dots, n-1$

The motion of the system associated to the car is then characterized by the equation:

$$\dot{q} = w_n X_n^1(q) + v_n X_n^2(q)$$

It is a controlled system with controls  $v_n$  and  $w_n$ , ( $v_n$  is the tangential velocity and  $w_n$  is the angular velocity as we have already seen at the beginning of the section). Each trajectory of the kinematic evolution of the car towing  $n$  trailers is an integral curve of the 2-distribution, on  $\mathbb{R}^2 \times (\mathbb{S}^1)^{n+1}$ ,

generated by:

$$\begin{cases} X_n^1 &= \frac{\partial}{\partial \theta_n} \\ X_n^2 &= \cos \theta_0 f_0^n \frac{\partial}{\partial x} + \sin \theta_0 f_0^n \frac{\partial}{\partial y} + \sin(\theta_1 - \theta_0) f_1^n \frac{\partial}{\partial \theta_0} + \cdots + \sin(\theta_n - \theta_{n-1}) \frac{\partial}{\partial \theta_{n-1}} \end{cases}$$

The distribution generated by  $\{X_1^n, X_2^n\}$ , naturally associated to the system of the car with  $n$  trailers, is a Goursat distribution.

### 3. Articulated arm

The purpose of this section is to construct a distribution  $\Delta$ , of dimension  $k+1$ , naturally associated to an  $(n+1)$  articulated arm, and which generates a special  $k$ -flag of length  $(n+1)$  on the configuration space  $\mathcal{C} \equiv \mathbb{R}^{k+1} \times (\mathbb{S}^k)^{n+1}$ . Moreover, the kinematic evolution of this arm is an integral curve of  $\Delta_n$ . We begin by recalling the context of special multi-flag in the formalism of [1], [14].

#### 3.1. Special multi-flags.

A special  $k$ -flag of length  $s$  is a sequence

$$D = D^s \subset D^{s-1} \subset \cdots \subset D^j \subset \cdots \subset D^1 \subset D^0 = TM$$

of distributions on a manifold  $M$  of dimension  $(s+1)k+1$  which satisfies the following conditions:

- (i)  $D^{j-1} = [D^j, D^j]$
- (ii)  $D^s, D^{s-1}, \dots, D^j, \dots, D^1, D^0$  are of respective ranks  $k+1, 2k+1, \dots, sk+1, (s+1)k+1$ .
- (iii) Each Cauchy characteristic subdistribution  $L(D^j)$  of  $D^j$  is a subdistribution of constant corank one in each  $D^{j+1}$ , for  $j = 1, \dots, s-1$ , and  $L(D^s) = 0$ .
- (iv) there exists a completely integrable subdistribution  $F \subset D^1$  of corank one in  $D^1$ .

**Remark 3.1.** It should be remarked that the covariant subdistribution  $F \subset D^1$  is uniquely determined by  $D^1$  itself. This covariant subdistribution  $F$  is completely described in [1] and [24] where it is defined in terms of the annihilating Pfaffian system  $(D^1)^\perp \subset T^*M$  ([7]). For a complete clarification on this fact see [16].

**Remark 3.2.** In the following we mean by a special multi-flag distribution all distribution generating a special multi-flag.

From the definition above, we obtain the following sandwich diagram:

$$\begin{array}{ccccccc} D^s & \subset & D^{s-1} & \subset \cdots \subset & D^j & \subset \cdots \subset & D^1 \subset D^0 = TM \\ \cup & & \cup & & \cdots & & \cup \\ L(D^{s-1}) & \subset & L(D^{s-2}) & \subset \cdots \subset & L(D^{j-1}) & \subset \cdots \subset & F \end{array}$$

All vertical inclusions in this diagram are of codimension one, while all horizontal inclusions are of codimension  $k$ . The squares built by these inclusions can be perceived as certain sandwiches, i.e each “subdiagram” number  $j$  indexed by the upper left vertices  $D^j$ :

$$\begin{array}{ccc} D^j & \subset & D^{j-1} \\ \cup & & \cup \\ L(D^{j-1}) & \subset & L(D^{j-2}) \end{array}$$

is called sandwich number  $j$ .

We can read the length  $s$  of the special  $k$ -flag by adding one to the total number of sandwiches in the sandwich diagram.

**Remark 3.3.** In a sandwich number  $j$ , at each point  $x \in M$ , in the  $(k+1)$  dimensional vector space  $D^{j-1}/L(D^{j-1})(x)$  we can look for the relative position of the  $k$  dimensional subspace  $L(D^{j-2})/L(D^{j-1})(x)$  and the 1 dimensional subspace  $D^j/L(D^{j-1})(x)$ :

either  $L(D^{j-2})/L(D^{j-1})(x) \oplus D^j/L(D^{j-1})(x) = D^{j-1}/L(D^{j-1})(x)$   
or  $D^j/L(D^{j-1})(x) \subset L(D^{j-2})/L(D^{j-1})(x)$ .

We say that  $x \in M$  is a **regular point** if the first situation is true in each sandwich number  $j$ , for  $j = 1, \dots, s$ . Otherwise  $x$  is called a **singular point**.

The set of singular points in the context of an articulated arm is studied in [25] and these results will be published in a future paper.

### 3.2. Special multi-flags and articulated arm.

The space  $(\mathbb{R}^{k+1})^{n+2}$ , will be written as the product  $\mathbb{R}_0^{k+1} \times \dots \times \mathbb{R}_i^{k+1} \times \dots \times \mathbb{R}_{n+1}^{k+1}$ . Let  $x_i = (x_i^1, \dots, x_i^{k+1})$  be the canonical coordinates on the space  $\mathbb{R}_i^{k+1}$  which is equipped with its canonical scalar product  $<, >$ .  $(\mathbb{R}^{k+1})^{n+2}$  is equipped with its canonical scalar product too.

Consider an articulated arm of length  $(n+1)$  denoted by  $(M_0, \dots, M_{n+1})$ . In this paper, we assume that the distances  $l_i$  are all equal to 1. On  $(\mathbb{R}^{k+1})^{n+2}$ , consider the vector fields:

$$(6) \quad \mathcal{Z}_i = \sum_{r=1}^{k+1} (x_{i+1}^r - x_i^r) \frac{\partial}{\partial x_i^r} \text{ for } i = 0, \dots, n$$

From our previous assumptions (see section 1), the kinematic evolution of the articulated arm is described by a controlled system:

$$(7) \quad \dot{q} = \sum_{i=0}^n u_i \mathcal{Z}_i + \sum_{r=1}^{k+1} u_{n+r} \frac{\partial}{\partial x_{n+1}^r}$$

with the following constraints:

$\|x_i - x_{i+1}\| = 1$  for  $i = 0 \dots n$  (see [10] chapter 2).

Consider the map  $\Psi_i(x_0, \dots, x_{n+1}) = \|x_i - x_{i+1}\|^2 - 1$ . Then, the configuration space  $\mathcal{C}$  is the set

$$(8) \quad \{(x_0, \dots, x_{n+1}), \text{ such that } \Psi_i(x_0, \dots, x_{n+1}) = 0 \text{ for } i = 0, \dots, n\}$$

For  $i = 0, \dots, n$ , the vector field:

$$(9) \quad \mathcal{N}_i = \sum_{r=1}^{k+1} (x_{i+1}^r - x_i^r) \left[ \frac{\partial}{\partial x_{i+1}^r} - \frac{\partial}{\partial x_i^r} \right]$$

is proportional to the gradient of  $\Psi_i$ . So the tangent space  $T_q \mathcal{C}$  is the subspace of  $T_q(\mathbb{R}^{k+1})^{n+2}$  which is orthogonal to  $\mathcal{N}_i(q)$  for  $i = 0, \dots, n$ .

Denote by  $\mathcal{E}$  the distribution generated by the vector fields

$$\{\mathcal{Z}_0, \dots, \mathcal{Z}_n, \frac{\partial}{\partial x_{n+1}^1}, \dots, \frac{\partial}{\partial x_{n+1}^{k+1}}\}.$$

**Lemma 3.1.** Let  $\Delta$  be the distribution on  $\mathcal{C}$  defined by  $\Delta(q) = T_q \mathcal{C} \cap \mathcal{E}$ . Then  $\Delta$  is a distribution of dimension  $k+1$  generated by

$$(x_{n+1}^r - x_n^r) \left[ \sum_{i=0}^n \prod_{j=i+1}^{n+1} A_j \mathcal{Z}_i \right] + \frac{\partial}{\partial x_{n+1}^r} \text{ for } r = 1 \dots k+1$$

where  $A_j(q) = -\langle \mathcal{N}_j(q), \mathcal{N}_{j-1}(q) \rangle = \langle \mathcal{Z}_j(q), \mathcal{N}_{j-1}(q) \rangle$  for  $j = 1, \dots, n$  and  $A_{n+1} = 1$ .

*Proof.* Any vector field  $X$  tangent to  $\mathcal{E}$  can be written as:

$$X = \sum_{i=0}^n \lambda_i \mathcal{Z}_i + \sum_{r=1}^{k+1} \mu_r \frac{\partial}{\partial x_{n+1}^r}$$

On the other hand, on  $\mathcal{C}$ , a vector fields  $X$  is tangent to  $\mathcal{C}$ , if and only if  $X$  is orthogonal to the vector fields  $\mathcal{N}_0, \dots, \mathcal{N}_n$ .

For  $i = 0, \dots, n-1$ , each relation  $\langle X, \mathcal{N}_i \rangle = 0$  is reduced to  $\langle \lambda_{i+1} \mathcal{Z}_{i+1} + \lambda_i \mathcal{Z}_i, \mathcal{N}_i \rangle = 0$ , which is equivalent to

$$(10) \quad \lambda_i = \lambda_{i+1} A_{i+1}$$

Similarly, the relation  $\langle X, \mathcal{N}_n \rangle = 0$  induces

$$(11) \quad \lambda_n = \sum_{r=1}^{k+1} \mu_r (x_{n+1}^r - x_n^r)$$

and from (10) and (11) we get  $\lambda_i = \prod_{j=i+1}^n A_j \lambda_n$ , for  $i = 0, \dots, n-1$ .

□

The properties of  $\Delta$  are summarized in the following result. (see also [10] chapter 2))

**Theorem 3.1.** *On  $\mathcal{C}$ , the distribution  $\Delta$  satisfies the following properties:*

- (1)  $\Delta$  is a distribution of rank  $k+1$ .
- (2) The distribution  $\Delta$  is a special  $k$ -flag on  $\mathcal{C}$  of length  $(n+1)$ .

The first part of Theorem 3.1 is a direct consequence of Lemma 3.1. Part (2) will be proved in section 6 in terms of hyperspherical coordinates.

#### 4. The evolution of the articulated arm in a system of angular coordinates

Given an articulated arm  $(M_0, \dots, M_{n+1})$  in  $\mathbb{R}^{k+1}$ , we will show that the constraint controlled system (7) can be written in the same way as (5) in an adapted system of angular coordinates with  $(k+1)$  controls, namely  $v_n$  (the "normal" velocity of  $M_{n+1}$ ) and the  $k$  components of the "tangential velocity" of  $M_{n+1}$  (Theorem 4.1).

##### 4.1. Hyperspherical coordinates.

The following map

$$\Gamma(x_0, x_1, \dots, x_i, \dots, x_{n+1}) = (x_0, x_1 - x_0, \dots, x_i - x_{i-1}, \dots, x_{n+1} - x_n)$$

implies a global diffeomorphism of  $(\mathbb{R}^{k+1})^{n+2}$  into itself and  $\Gamma(\mathcal{C}) = \mathbb{R}^{k+1} \times (\mathbb{S}^k)^{n+1}$  where  $\mathbb{S}^k$  is the canonical sphere in  $\mathbb{R}^{k+1}$ . In this representation, the canonical coordinates on  $(\mathbb{R}^{k+1})^{n+2} = \Gamma((\mathbb{R}^{k+1})^{n+2})$  will be denoted by  $(x_0, z_1, \dots, z_i, \dots, z_{n+1})$  so that  $\Gamma$  is given by  $x_0 = x_0$  and  $z_i = x_{i+1} - x_i$  for  $i = 0, \dots, n$ . Via this global chart, each point  $q = (x_0, x_1, \dots, x_i, \dots, x_{n+1}) \in \mathcal{C}$  can be identified with  $(x_0, z_1, \dots, z_i, \dots, z_{n+1}) \in \mathbb{R}^{k+1} \times (\mathbb{S}^k)^{n+1}$  for  $i = 0, \dots, n$  and  $\mathcal{C}$  can be identified with  $\mathcal{S} = \mathbb{R}^{k+1} \times (\mathbb{S}^k)^{n+1}$ .

We will put on  $(\mathbb{S}^k)^{n+1}$  charts given by *hyperspherical coordinates*. We first recall some basic facts about this type of coordinates.

The *hyperspherical coordinates* in  $\mathbb{R}^{k+1}$  are given by the relations:

$$(12) \quad \begin{cases} z^1 = \rho\phi^1(\theta) = \rho \sin \theta^1 \cdots \sin \theta^{k-1} \sin \theta^k \\ z^2 = \rho\phi^2(\theta) = \rho \sin \theta^1 \cdots \sin \theta^{k-1} \cos \theta^k \\ z^3 = \rho\phi^3(\theta) = \rho \sin \theta^1 \cdots \sin \theta^{k-2} \cos \theta^{k-1} \\ \cdots \\ z^k = \rho\phi^k(\theta) = \rho \sin \theta^1 \cos \theta^2 \\ z^{k+1} = \rho\phi^{k+1}(\theta) = \rho \cos \theta^1 \end{cases}$$

with  $\rho^2 = (z^1)^2 + \cdots + (z^{k+1})^2$ ,  $0 \leq \theta^k \leq 2\pi$  and  $0 \leq \theta^j \leq \pi$  for  $1 \leq j \leq k-1$ .

We consider  $\hat{\Phi}(\rho, \theta) = \rho\Phi(\theta) = z$ , the application from  $]0, +\infty[ \times [0, \pi] \times \cdots \times [0, \pi] \times [0, 2\pi]$  to  $\mathbb{R}^{k+1}$ .

**Remark 4.1.** *The previous expression uses the "geographical" version of hyperspherical coordinates. An another version, maybe more usual, can be obtained by taking  $\frac{\pi}{2} - \theta^k$  instead of  $\theta^k$  and then, permuting the functions sine and cosine in each formula. However, our choice is motivated by the following fact: the evolution of the articulated arm of length  $(n+1)$  written in a such chart, (see (18)) gives exactly the system (5) for  $n=1$ .*

The jacobian matrix  $D\hat{\Phi}$  of  $\hat{\Phi}$  is:

$$\begin{pmatrix} \sin \theta^1 \cdots \sin \theta^{k-1} \sin \theta^k & \rho \cos \theta^1 \cdots \sin \theta^{k-1} \sin \theta^k & \cdots & \rho \sin \theta^1 \cdots \sin \theta^{k-1} \cos \theta^k \\ \sin \theta^1 \cdots \sin \theta^{k-1} \cos \theta^k & \rho \cos \theta^1 \cdots \sin \theta^{k-1} \cos \theta^k & \cdots & -\rho \sin \theta^1 \cdots \sin \theta^{k-1} \sin \theta^k \\ \cdots & \cdots & \cdots & \cdots \\ \cos \theta^1 & -\rho \sin \theta^1 & 0 & 0 \end{pmatrix}$$

It is well known that  $\det(D\hat{\Phi})(\rho, \theta) = (-1)^{[k+1/2]}(\rho)^k \prod_{i=1}^{k-1} (\sin \theta^{k-i})^i$ . It follows that  $D\hat{\Phi}$  is invertible only for  $0 \leq \theta^k \leq 2\pi$  and  $0 < \theta^j < \pi$  for  $j = 1, \dots, k-1$ .

In the sequence we note  $\{\nu, \Theta^1, \dots, \Theta^k\}$  the moving frame on  $\hat{\Phi}(]0, +\infty[ \times [0, \pi] \times \cdots \times [0, \pi] \times [0, 2\pi])$  that is the image, by  $D\hat{\Phi}$ , of the canonical frame  $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta^1}, \dots, \frac{\partial}{\partial \theta^k}\}$ .

Consider a point  $z = \Phi(\theta) = \hat{\Phi}(1, \theta)$ . We note that, in this case, we have

$$D\hat{\Phi} = \begin{pmatrix} \phi & \frac{\partial \phi}{\partial \theta^1} & \cdots & \frac{\partial \phi}{\partial \theta^k} \end{pmatrix}$$

where  $\phi$  (resp.  $\frac{\partial \phi}{\partial \theta^j}$ ) is the column vector of components  $\{\phi^1, \dots, \phi^{k+1}\}$  (resp.  $\{\frac{\partial \phi^1}{\partial \theta^j}, \dots, \frac{\partial \phi^{k+1}}{\partial \theta^j}\}$ ).

Note that, in canonical coordinates, this moving frame  $\{\nu, \Theta^1, \dots, \Theta^k\}$  can be written

$$(13) \quad \nu(z) = \sum_{r=1}^{k+1} \phi^r \frac{\partial}{\partial x^r} \quad \text{and} \quad \Theta^j(z) = \sum_{r=1}^{k+1} \frac{\partial \phi^r}{\partial \theta^j} \frac{\partial}{\partial x^r}$$

and in hyperspherical coordinates this moving frame is:

$$(14) \quad \nu(\theta) = \frac{\partial}{\partial t} \quad \text{and} \quad \Theta^j(\theta) = \frac{\partial}{\partial \theta^j}$$

On the other hand, it is well known that  $\det(D\hat{\Phi})(\rho, \theta) = (-1)^{[k+1/2]}(\rho)^k \prod_{i=1}^{k-1} (\sin \theta^{k-i})^i$ . It follows that  $D\hat{\Phi}$  is invertible only for  $0 \leq \theta^k \leq 2\pi$  and  $0 < \theta^j < \pi$  for  $j = 1, \dots, k-1$ . The column vectors of the jacobian matrix  $D\hat{\Phi}$  are pairwise orthogonal and we have



$\|\phi\|^2 = \|\frac{\partial\phi}{\partial\theta^1}\|^2 = 1$ ,  $\|\frac{\partial\phi}{\partial\theta^j}\|^2 = (\sin\theta^1 \cdots \sin\theta^{j-1})^2$  for  $j = 2, \dots, k$ .

So, at each point  $z = \Phi(\theta)$ , the inverse of this jacobian matrix is the transpose of the matrix:

$$(15) \quad \begin{pmatrix} \phi & \frac{1}{\|\frac{\partial\phi}{\partial\theta^1}\|} \frac{\partial\phi}{\partial\theta^1} & \cdots & \frac{1}{\|\frac{\partial\phi}{\partial\theta^k}\|} \frac{\partial\phi}{\partial\theta^k} \end{pmatrix}$$

This inverse exists only if  $0 \leq \theta^k \leq 2\pi$  and  $0 < \theta^j < \pi$  for  $j = 1, \dots, k-1$ .

Consider a hyperspherical chart  $y = \hat{\Phi}(\rho, \theta) + a$  around some  $a \in \mathbb{R}^{k+1}$  and note  $\{\nu, \Theta^1, \dots, \Theta^k\}$  its associated moving frame.

Let  $y' = \hat{\Phi}'(\rho', \theta') + a'$  be another hyperspherical chart around some other point  $a' \in \mathbb{R}^{k+1}$  such that its domain intersects the domain of  $\hat{\Phi}$ . Note  $\{\nu', \Theta'^1, \dots, \Theta'^k\}$  the moving frame associated to it. So, at each point of the sphere  $\mathbb{S}_a$  of center  $a$  of radius 1 in  $\mathbb{R}^{k+1}$  contained in the intersection of these domains, we can write :

$$(16) \quad \nu' = A\nu + \sum_{j=1}^k B^j \Theta^j$$

The components of these vectors are actually the components of the first column vector of the matrix  $[D\hat{\Phi}]^{-1} \circ D\hat{\Phi}'$ . According to (15), we get:

$$(17) \quad \begin{cases} \bullet A(\theta, \theta') = \sum_{r=1}^{k+1} \phi^r \phi'^r \\ \bullet B^1(\theta, \theta') = \sum_{r=1}^{k+1} \frac{\partial\phi^r}{\partial\theta^1} \phi'^r \\ \bullet B^j(\theta, \theta') = \frac{1}{\|\frac{\partial\phi}{\partial\theta^j}\|} \sum_{r=1}^{k+1} \frac{\partial\phi^r}{\partial\theta^j} \phi'^r \end{cases}$$

**Remark 4.2.**

the vector  $\sum_{j=1}^k B^j \Theta^j$  is nothing but the orthogonal projection of  $\nu'$  on to  $T\mathbb{S}_a$ .

#### 4.2. The evolution of the articulated arm on $\mathcal{S}$ .

Coming back to  $\mathcal{S} = \mathbb{R}^{k+1} \times (\mathbb{S}^k)^{n+1}$  which is considered as a subset in  $(\mathbb{R}^{k+1})^{n+2}$ , let  $\mathbb{S}_i$ ,  $i = 0, \dots, n$ , be the canonical sphere in  $\mathbb{R}_{i+1}^{k+1}$ . Recall that the canonical coordinates on  $\mathbb{R}_{i+1}^{k+1}$  are denoted by  $z_i = (z_i^1, \dots, z_i^r, \dots, z_i^{k+1})$ . Given a point  $\alpha$  in the sphere  $\mathbb{S}_i$ , there exists a hyperspherical chart  $z_{i+1} = \hat{\Phi}_i(\rho_i, \theta_i) = \rho_i \Phi_i(\theta_i^1, \dots, \theta_i^k)$  defined for  $0 \leq \theta_i^k \leq 2\pi$  and  $0 < \theta_i^j < \pi$ ,  $j = 1, \dots, k-1$ , where  $\Phi_i(0, \dots, 0) = \alpha$ . So, for a given point  $q = (x_0, z_1, \dots, z_i, \dots, z_{n+1}) \in \mathcal{S}$ , we get a chart  $(Id - x_0, \hat{\Phi}_0, \dots, \hat{\Phi}_1, \dots, \hat{\Phi}_n)$  centered at  $q$ , such that its restriction to  $\rho_i = 1$ ,  $i = 0, \dots, n$ , induces a chart of  $\mathcal{S}$  (centered at  $q$ ). For  $i = 0, \dots, n$ , in a neighborhood of each  $z_{i+1} \in \mathbb{R}_{i+1}^{k+1}$ , we consider the moving frame

$$\mathcal{R}_i = \{\nu_i, \Theta_i^1, \dots, \Theta_i^k\}.$$

**Remark 4.3.**

- (1) Given  $q = (x_0, \dots, x_{n+1}) \in \mathcal{C}$ , for  $i = 0, \dots, n$ , denote by  $\tilde{\mathbb{S}}_i$  the sphere in  $\mathbb{R}^{k+1}$  of center  $x_i$  and radius 1. One can put on  $\mathbb{R}^{k+1}$  the hyperspherical coordinates  $y_i = \hat{\Phi}_i(\rho_i, \theta_i) + x_i$ . As  $x_{i+1}$  belongs to  $\tilde{\mathbb{S}}_i$ , on a neighborhood of  $x_{i+1}$ , we have also the following moving frame (again denoted  $\mathcal{R}_i$ ):

$$\mathcal{R}_i = \{\nu_i, \Theta_i^1, \dots, \Theta_i^k\}.$$

Note that on  $x_{i+1}$ , the outward normal unit vector of  $\tilde{\mathbb{S}}_i$  is  $\nu_i(x_{i+1})$ , and  $\{\Theta_i^1(x_{i+1}), \dots, \Theta_i^k(x_{i+1})\}$  is a basis  $T_{x_{i+1}}\tilde{\mathbb{S}}_i$ .

- (2) From part (1), if  $\zeta = \Gamma(q)$ , we get an isomorphism from  $T_\zeta \mathcal{S}$  to  $T_{x_0} \mathbb{R}^{m+1} \oplus T_{x_1} \tilde{\mathcal{S}}_0 \oplus \cdots \oplus T_{x_{n+1}} \tilde{\mathcal{S}}_n$ .  
So we can identify  $T_\zeta \mathcal{S}$  with  $T_{x_0} \mathbb{R}^{n+1} \oplus T_{x_1} \tilde{\mathcal{S}}_0 \oplus \cdots \oplus T_{x_{n+1}} \tilde{\mathcal{S}}_n$ .

**Notations 4.1.**

in hyperspherical coordinates, we define on  $\mathcal{S}$ :

- (1)  $A_i = \sum_{r=1}^{k+1} \phi_{i-1}^r \phi_i^r$  for  $i = 1, \dots, n$  and  $A_{n+1} = 1$ ;
- (2)  $Z_0 = \sum_{r=1}^{k+1} \phi_0^r \frac{\partial}{\partial x^r}$
- (3)  $Z_i = \sum_{j=1}^k B_i^j \frac{\partial}{\partial \theta_{i-1}^j}$  for  $i = 1, \dots, n$   
with:
  - $B_i^1 = \sum_{r=1}^{k+1} \frac{\partial \phi_{i-1}^r}{\partial \theta_{i-1}^1} \phi_i^r$  for  $i = 1, \dots, n$
  - $B_i^j = \frac{1}{\|\frac{\partial \phi_{i-1}}{\partial \theta_{i-1}^j}\|} \sum_{r=1}^{k+1} \frac{\partial \phi_{i-1}^r}{\partial \theta_{i-1}^j} \phi_i^r$  for  $i = 1, \dots, n$  and  $j = 2, \dots, k$
- (4)  $X_m^i = \frac{\partial}{\partial \theta_m^i}$ , for  $i = 1, \dots, k$ ,  $m = 0, \dots, n$
- (5)  $X_m^0 = \sum_{i=0}^m f_m^i Z_i$ , for  $m = 0, \dots, n$   
with  $f_m^r = \prod_{j=r+1}^m A_j$ , for  $r = 0, \dots, m-1$ , and  $f_m^m = 1$  for  $m = 0, \dots, n$ .
- (6)  $\Delta_n$  the distribution generated by  $\{X_n^0, X_n^1, \dots, X_n^k\}$  (with previous notations).

**Remark 4.4.**

- (1) For  $i = 0, \dots, n$ , consider  $[\Phi_i]$  the column matrix of components  $(\phi_i^1, \dots, \phi_i^{k+1})$  and  $[D\Phi_i]^{-1}$  the matrix composed by the last  $k$  rows of the jacobian matrix of the application  $(\Phi_i)^{-1}$ .  
Finally, denote by  $[\frac{\partial}{\partial \theta_i}]$  (resp.  $[\frac{\partial}{\partial x}]$ ) the column matrix of components  $(\frac{\partial}{\partial \theta_i^1}, \dots, \frac{\partial}{\partial \theta_i^k})$ ,  
(resp.  $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{k+1}})$ ) for  $i = 0, \dots, n$ . So we can write:  
$$Z_0 = {}^t [\frac{\partial}{\partial x}] [\Phi_0] \text{ and } Z_i = {}^t [\frac{\partial}{\partial \theta_{i-1}}] [[D\Phi_{i-1}]]^{-1} [\Phi_i], \text{ for } i = 1, \dots, n.$$
- (2) According to Remark 4.3 part (2), in the identification :  
$$T_\zeta \mathcal{S} \equiv T_{x_0} \mathbb{R}^{n+1} \oplus T_{x_1} \tilde{\mathcal{S}}_0 \oplus \cdots \oplus T_{x_{n+1}} \tilde{\mathcal{S}}_n,$$
  
the component of  $X_n^0$  on  $T_{x_i} \tilde{\mathcal{S}}_{i-1}$  is exactly  $f_n^i Z_i(\zeta)$ .

With the previous notations we have the following result

**Theorem 4.1.**

- (1) On  $\mathcal{S}$ , the distribution  $\Delta_n$  is the image by  $\Gamma$  of the distribution  $\Delta$  where  $\Gamma : \mathcal{C} \rightarrow \mathcal{S}$  is the diffeomorphism defined at the beginning of Subsection 4.1.

- (2) The evolution of the articulated arm of length  $(n+1)$  is described in a chart, by the following controlled system with  $k+1$  controls :

$$(18) \quad \left\{ \begin{array}{l} \dot{x}^1 = v_0 \phi_0^1 \\ \dot{x}^2 = v_0 \phi_0^2 \\ \dots \\ \dot{x}^{k+1} = v_0 \phi_0^{k+1} \\ \dot{\theta}_0^1 = v_1 B_1^1 \\ \dots \\ \dot{\theta}_0^k = v_1 B_1^k \\ \dots \\ \dot{\theta}_i^1 = v_{i+1} B_{i+1}^1 \\ \dots \\ \dot{\theta}_i^k = v_{i+1} B_{i+1}^k \\ \dots \\ \dot{\theta}_{n-1}^1 = v_n B_n^1 \\ \dots \\ \dot{\theta}_{n-1}^k = v_n B_n^k \\ \dot{\theta}_n^1 = v_{\theta_n^1} \\ \dots \\ \dot{\theta}_n^k = v_{\theta_n^k} \end{array} \right.$$

where  $v_i = v_n \prod_{r=i+1}^n A_r$  and  $(v_{\theta_n^1}, \dots, v_{\theta_n^k}, v_n)$  are the  $(k+1)$  controls of the system (18).

Moreover, according to Remark 4.3 we have:

- $(v_{\theta_n^1}, \dots, v_{\theta_n^k})$  are the "tangential components" of the velocity of  $M_{n+1}$ , namely the components, in the canonical basis of  $T_{x_{n+1}} \tilde{S}_n$ , of the orthogonal projection of the velocity of  $M_{n+1}$ ;
- $v_{i-1}$  is the "normal velocity" of  $M_i$  for all  $i = 1, \dots, n+1$ , namely the components of the orthogonal projection of the velocity of  $M_i$  on the direction generated by  $\nu_{i-1}(x_i)$ .

**Remark 4.5.** Equations of system (18), for  $k = 1$ , are exactly (with the same notations) the classical modeling of the car with  $n$  trailers ([2], [3], [8], [19], [26])

**Remark 4.6.** According to Remark 3.3, for  $k \geq 2$ , a point  $q = (x_0, z_1, \dots, z_{n+1})$  is singular if and only if there exists an index  $0 \leq i \leq n$  such that  $A_i(q) = 0$  which is equivalent to  $[M_{i-1}, M_i]$  and  $[M_i, M_{i+1}]$  are orthogonal in  $M_i$ . In this situation, the velocity all  $M_j$  points are zero, for  $j < i$ . The set of such points is studied in [25].

## 5. PROOF OF THEOREM 4.1

For part (1), it is sufficient to prove that any germ curve  $\gamma$  in  $\mathcal{C}$  is tangent to  $\Delta$  at a point  $q \in \mathcal{C}$ , if and only if,  $\Gamma \circ \gamma$  is tangent to the distribution generated by  $\{X_n^0, X_n^1, \dots, X_n^k\}$  at  $\Gamma(q)$ .

Let be  $\gamma(t) = (x_0(t), x_1(t), \dots, x_{n+1}(t))$  a curve in  $\mathcal{C}$  defined on  $] -\varepsilon, \varepsilon[$  with  $\gamma(0) = q$ . We have

$$(19) \quad x_i = x_{i-1} + z_i \text{ for } i = 1 \dots n+1$$

Assume that  $\gamma$  is tangent to  $\Delta$ . It follows that we have for each  $t$  :

$$(20) \quad \dot{x}_{i-1} = v_{i-1} z_i, \text{ for } i = 1, \dots, n+1,$$

In view of (20), by differentiation of (19) we get

$$(21) \quad \dot{x}_i = v_{i-1} z_i + \dot{z}_i \text{ for } i = 1, \dots, n+1$$

for some  $v_i \in \mathbb{R}$  (which depends of  $t$ ).

On the other hand, in  $\mathbb{R}^{k+1}$  we also have the orthogonal decomposition

$$(22) \quad z_{i+1} = \langle z_{i+1}, z_i \rangle z_i + \tilde{z}_i \quad i = 0, \dots, n.$$

where

$$(23) \quad \tilde{z}_i \text{ is the othogonal projection of } z_{i+1} \text{ on to the hyperplan orthogonal to } z_i.$$

So we get:

$$(24) \quad \dot{x}_i = v_i z_{i+1} = v_i \langle z_{i+1}, z_i \rangle z_i + v_i \tilde{z}_i \text{ for } i = 1, \dots, n$$

Comparing (21) and (24), we get :

$$(25) \quad v_{i-1} = v_i \langle z_{i+1}, z_i \rangle \text{ and } \dot{z}_i = v_i \tilde{z}_i \text{ for } i = 1, \dots, n$$

- for  $i = n + 1$ , we have the orthogonal decomposition

$$(26) \quad \dot{x}_{n+1} = v_{n+1} z_{n+1} + \tilde{z}_{n+1}$$

But we also have

$$\dot{x}_{n+1} = \dot{z}_{n+1} + \dot{x}_n = v_n z_{n+1} + \dot{z}_{n+1}$$

So from the unicity of the orthogonal decomposition, we get

$$(27) \quad v_n = v_{n+1} \text{ and } \dot{z}_{n+1} = \tilde{z}_{n+1}$$

- for  $i = 0$  we have of course

$$(28) \quad \dot{x}_0 = v_0 z_1$$

From (25) and (27) we get the relation

$$(29) \quad v_i = \prod_{j=i+2}^n \langle z_{j+1}, z_j \rangle v_n \text{ for } i = 0 \dots, n-1$$

On the other hand, according to Remark 4.3 part (2), as  $x_i \in \tilde{\mathbb{S}}_{i-1}$ , for  $t = 0$ ,  $\dot{x}_i$  can be considered as a vector in  $T_{x_i} \mathbb{R}^{k+1}$  and so, in the moving frame  $\mathcal{R}_i$ , the relations (24), (29) and (26) give rise to the following decompositions

$$(30) \quad \dot{x}_i = v_{i-1} \nu_{i-1}(x_i) + v_i Z_i \text{ for } i = 1, \dots, n, \text{ and } \dot{x}_{n+1} = v_n \nu_n(x_{n+1}) + \tilde{Z}_{n+1}$$

where  $Z_i$  (resp.  $\tilde{Z}_{n+1}$ ) belongs to  $T_{x_i} \tilde{\mathbb{S}}_{i-1}$  pour  $i = 1, \dots, n$  (resp.  $T_{x_{n+1}} \tilde{\mathbb{S}}_n$ ).

In fact, according to (23) and Remark 4.3 part (2),  $Z_i$  is the orthogonal projection of  $(D\hat{\Phi}_{i-1}^{-1})^{-1} \circ D\hat{\Phi}_i(\nu_i(x_{i+1}))$  on  $T_{x_i} \tilde{\mathbb{S}}_{i-1}$ , for  $i = 1, \dots, n$ . Taking in account Remark 4.2 and (17), we get the expression of  $Z_i$  given in Notations 4.1 for  $i = 1, \dots, n$ . Moreover, the  $i^e$  component  $\dot{z}_i$  of  $D\Gamma(\dot{\gamma}(0))$  is  $v_{i-1} Z_i$  in hyperspherical coordinates for  $i = 1, \dots, n$ . On the other hand, in the hyperspherical  $\Phi_n$ , the vector  $\tilde{Z}_{n+1}$  can be written

$$\tilde{Z}_{n+1} = \sum_{j=1}^k w_j \frac{\partial}{\partial \theta_n^j}$$

Finally, from (29) and the value of  $A_j$ , we get:

$$v_i = f_n^i v_n \text{ for } i = 0, \dots, n$$

On the other hand, from (28), as vector in  $T_{x_0} \mathbb{R}^{m+1}$ , we have the following decomposition:

$$\dot{x}_0 = v_0 \sum_{r=1}^m \phi_0^r \frac{\partial}{\partial x_0^r}$$

So, according to Remark 4.3 part (2) and Remark 4.4 part (2), in hyperspherical coordinates, we finally obtain that,  $D\Gamma(\dot{\gamma}(0))$  is tangent  $\Delta_n$  at  $\zeta = \Gamma(q)$ .

Conversely, let be  $\delta(t) = (x_0(t), z_1(t), \dots, z_{n+1}(t))$  a curve in  $\mathcal{S}$ , defined on  $] -\varepsilon, \varepsilon[$ , and such that  $\delta(0) = \zeta = \Gamma(q)$ . Assume that, in hyperpspherical coordinates,  $\delta$  is tangent to  $\Delta_n$ . and so  $\delta$  satisfies the differential equation (18). According to the definitions of  $Z_i$  in hyperpspherical coordinates and taking in account (23) and Remark 4.3 part (2), there exists a curve  $v_n(t)$  in  $\mathbb{R}$  such that, for any  $i = 1, \dots, n$ , we have

$$\dot{z}_i = [(\prod_{j=i+1}^n < z_{j+1}, z_j >) v_n] [z_{i+1} - < z_{i+1}, z_i > z_i] \text{ for } i = 1, \dots, n$$

and

$$(31) \quad \dot{x}_0 = [(\prod_{j=1}^n < z_{j+1}, z_j >) v_h] z_1.$$

It follows that for  $i = 1, \dots, n$  we have:

$$(32) \quad \dot{x}_i = \dot{x}_0 + \sum_{j=1}^i \dot{z}_j = (\prod_{l=i+1}^n < z_{l+1}, z_l > v_n) z_{i+1}$$

so, according to (31) and (32), we obtain that  $\gamma(t) = \Gamma^{-1} \circ \delta(t)$  is a curve in  $\mathcal{C}$  which is tangent to  $\Delta$ .

Taking in account part (1) of Theorem 4.1, the kinematic evolution of the articulated arm is a controlled system on  $\mathcal{S}$  which is exactly (18). However, for the completeness of the proof of this result, we must prove the interpretation of the control in terms of the components of the velocity of  $M_i$ ,  $i = 1, \dots, n+1$ .

From (30), according to Remark 4.3 part (2), in hyperpspherical coordinates  $\Phi_n$  around  $x_{n+1} \in \mathbb{R}^{k+1}$ , the velocity  $\dot{x}_{n+1}$  of  $M_{n+1}$  can be written:

$$\dot{x}_{n+1} = v_n \nu_n(x_{n+1}) + \sum_{j=1}^k v_{\theta_n^j} \frac{\partial}{\partial \theta_n^j}$$

Where  $v_n$  is the "normal velocity" of  $M_{n+1}$  and  $(v_{\theta_n^1}, \dots, v_{\theta_n^k})$  are the "tangential components" of the velocity of  $M_{n+1}$ . Moreover, according to the value of  $X_n^0$ , the control parameters are exactly  $(v_n, v_{\theta_n^1}, \dots, v_{\theta_n^k})$ .

In the same way, for any  $i = 1, \dots, n$ , in hyperspherical coordinates  $\Phi_{i-1}$  around  $x_i \in \mathbb{R}^{k+1}$  the velocity  $\dot{x}_i$  of  $M_i$  has the decomposition

$$\dot{x}_i = v_{i-1} \nu_{i-1}(x_i) + v_i Z_i$$

where  $Z_i$  belongs to  $T_{x_i} \tilde{\mathcal{S}}_{i-1}$ . So,  $v_{i-1}$  the "normal velocity" of  $M_i$

## 6. PROOF OF THEOREM 3.1

We will see that the distribution  $\Delta_n$  generates actually a special  $k$ -flag of length  $(n+1)$  on a  $k(n+2)+1$  dimensional manifold. Let's introduce the following notations:

- $\Delta_m$  is the distribution generated by  $\{X_m^0, X_m^1, \dots, X_m^k\}$  for  $m = 1, \dots, n$ ;
- $D^{m+1}$  is the distribution generated by  $X_m^0$  and  $\{X_j^1, \dots, X_j^k \mid m \leq j \leq n\}$  for  $m = 0, \dots, n$ ;
- $D^0 = TM$
- $E^{m+1}$  is the distribution generated by  $\{X_j^1, \dots, X_j^k \mid m \leq j \leq n\}$  for  $m = 0, \dots, n$ .

**Proposition 6.1.**  $\Delta_n$  is a special  $k$ -flag distribution. More precisely, it satisfies the following properties:

- (1) For  $m = 1, \dots, n+1$ , the distributions  $D^m$  and  $E^m$ , are of respective constant dimensions  $(n-m+2)k+1$  and  $(n-m+2)k$ ;
- (2) for  $m = 1, \dots, n+1$ ,  $E^m$  is an involutive subdistribution of  $D^m$  of codimension 1. Moreover  $[E^{m+1}, D^{m+1}] \subset D^m$  for  $m = 1, \dots, n$ . Actually  $E^{m+1}$  is the "Cauchy-characteristic distribution" of  $D^m$  for  $m = 1, \dots, n$  ([13], [14]);
- (3)  $[D^{m+1}, D^{m+1}] = D^m$  for all  $m = 0, \dots, n$ ;
- (4)

$$\begin{array}{ccccccc} \Delta_n = D^{n+1} & \subset \dots \subset & D^m & \subset \dots \subset & D^1 & \subset & D^0 = TM \\ & \cup & & \cup & & \cup & \\ & E^{n+1} & \subset \dots \subset & E^m & \subset \dots \subset & E^1 & \end{array}$$

*Proof of Proposition 6.1.*

It is sufficient to show the property (4). The inclusions  $[E^{m+1}, D^{m+1}] \subset D^m$  for  $m = n, \dots, 0$ , are an easy consequence of (4) and the properties (1), (2) and (3) are always true, according to the definition of spaces  $E^m$ ,  $D^m$  and  $\Delta_m$ .

Denote by  $\Delta_0$  the distribution generated by  $\{\frac{\partial}{\partial x_0^1}, \dots, \frac{\partial}{\partial x_0^{k+1}}\}$ .

For all  $m = 1, \dots, n+1$ , we have :

$$D^m = E^{m+1} \oplus \Delta_{m-1} = D^{m+1} + \Delta_{m-1}.$$

$[D^{m+1}, D^{m+1}]$  contains the space generated by  $D^{m+1}$  and the Lie brackets  $[X_m^i, X_m^0]$ , for  $i = 1, \dots, k$ . We will show by induction that, in fact, they are generating  $[D^{m+1}, D^{m+1}]$  modulo  $D^{m+1}$ .

For all  $m = n, \dots, 0$ , we have  $X_m^0 = A_m X_{m-1}^0 + Z_m$ . It results from the definition of  $A_i$ ,  $Z_i$  and  $X_m^0$  that  $[X_m^i, X_{m-1}^0] = 0$ . So we have

$$[X_m^i, X_m^0] = X_m^i(A_m)X_{m-1}^0 + [X_m^i, Z_m].$$

For  $j = 1, \dots, k+1$ , consider the vector fields:

$$Y_{m-1}^j = \phi_{m-1}^j X_{m-1}^0 + \sum_{r=1}^k \frac{1}{\|\frac{\partial \phi_{m-1}}{\partial \theta_{m-1}^r}\|} \frac{\partial \phi_{m-1}^j}{\partial \theta_{m-1}^r} X_{m-1}^r$$

If we set  $\hat{\Phi}_{m-1}(\rho, \theta_{m-1}) = \rho \Phi_{m-1}(\theta_{m-1})$ , then we have the relation:

$$[Y_{m-1}] = [D\hat{\Phi}_{m-1}]^{-1}[X_{m-1}]$$

where the vectors column  $[Y_{m-1}]$  and  $[X_{m-1}]$  have  $\{Y_{m-1}^1, \dots, Y_{m-1}^{k+1}\}$  and  $\{X_{m-1}^0, \dots, X_{m-1}^k\}$  as components respectively. It results that  $\{Y_{m-1}^1, \dots, Y_{m-1}^{k+1}\}$  is a basis of  $\Delta_{m-1}$ .

For  $m = 0, \dots, n$ , we note  $[D\hat{\Phi}_m]$  the jacobian matrix of  $\hat{\Phi}_m(\rho, \theta_m) = \rho \Phi_m(\theta_m)$ .

The following decompositions occur:

$$X_m^0 = \sum_{j=1}^{k+1} \phi_m^j Y_{m-1}^j$$

$$[X_m^i, X_m^0] = \sum_{j=1}^{k+1} \frac{\partial \phi_m^j}{\partial \theta_m^i} Y_{m-1}^j \text{ for all } m = 1, \dots, n$$

By similar way for  $D\hat{\Phi}_m$ , we can show that the family of vector fields

$$\{X_m^0, [X_m^1, X_m^0], \dots, [X_m^k, X_m^0]\}$$

is also a basis of  $\Delta_{m-1}$ . This result is also true for  $m = 0$ .

Since  $D^{m+1} = E^{m+2} \oplus \Delta_m$ , the space  $[D^{m+1}, D^{m+1}]$  contains  $E^{m+2}$ , all vectors  $X_m^0, X_m^1, \dots, X_m^k$  and the Lie brackets  $[X_m^1, X_m^0], \dots, [X_m^k, X_m^0]$ . Also, all the Lie brackets  $[X_r^j, X_m^0]$

are zero for  $r = m + 1, \dots, n$  and  $j = 1, \dots, k$  since  $X_m^0$  does not depend on variables  $\theta_r^j$ , for  $r = m + 1, \dots, n$  and  $j = 1, \dots, k$ . The other Lie brackets  $[X_r^j, X_m^i]$  are zero for  $r = m + 1, \dots, n$  and  $i, j = 1, \dots, k$ .

Since  $\{X_m^0, [X_m^1, X_m^0], \dots, [X_m^k, X_m^0]\}$  is a basis of  $\Delta_{m-1}$ , then we have

$$[D^{m+1}, D^{m+1}] = E^{m+1} \oplus \Delta_{m-1} = D^m$$

which completes the proof of proposition 6.1 and Theorem 3.1.  $\square$

**Comment 6.1.**

Given two integers  $p$  and  $m$  such that  $1 \leq p < m \leq n$ , we can look for the motion of a "sub-induced arm", which consists of segments of the original arm between  $M_{p-1}$  and  $M_{m+1}$  included. We can then study the motion of  $M_{p-1}$  as the motion of the extremity of this sub arm for the motion commanded by the segment  $[M_m; M_{m+1}]$ . We put  $h = m - p + 1$ , and we write  $\Pi_{p,m}$  for the canonical projection from  $\mathbb{R}^{k+1} \times (\mathbb{S}^k)^{n+1}$  onto  $\mathbb{R}^{k+1} \times (\mathbb{S}^k)^{h+1}$  defined as  $\Pi_{p,m}(x, z_1, \dots, z_{n+1}) = (x_{p-1}, z_{p-1}, z_p, \dots, z_m)$  where  $x_{p-1}$  are the Cartesian coordinates of  $M_{p-1}$ .

The evolution of the extremity  $M_{p-1}$  of this articulated sub-arm, controlled by the movement of  $[M_m; M_{m+1}]$ , is a solution of the following differential system (with notations of Theorem 4.1):

$$(33) \quad \begin{cases} \dot{x}_{p-1}^1 = v_{p-1} \phi_{p-1}^1 \\ \dot{x}_{p-1}^2 = v_{p-1} \phi_{p-1}^2 \\ \dots \\ \dot{x}_{p-1}^{k+1} = v_{p-1} \phi_{p-1}^{k+1} \\ \dot{\theta}_{p-1}^1 = v_p B_p^1 \\ \dots \\ \dot{\theta}_{p-1}^k = v_p B_p^k \\ \dots \\ cr \dot{\theta}_i^1 = v_{i+1} B_{i+1}^1 \\ \dots \\ \dot{\theta}_i^k = v_{i+1} B_{i+1}^k \\ \dots \\ \dot{\theta}_{m-1}^1 = v_m B_m^1 \\ \dots \\ \dot{\theta}_{m-1}^k = v_m B_m^k \\ \dot{\theta}_m^1 = v_{\theta_m^1} = v_{m+1} B_{m+1}^1 \\ \dots \\ \dot{\theta}_m^k = v_{\theta_m^k} = v_{m+1} B_{m+1}^k \end{cases}$$

It is a controlled system on  $\mathbb{R}^{k+1} \times (\mathbb{S}^k)^{h+1}$  ( $h = m - p + 1$ ) :

$$\dot{\hat{q}} = u_0 \hat{X}_h^0 + \sum_{i=1}^k u_i X_h^i$$

with controls  $u_0 = v_m$ , and  $u_i = v_{\theta_m^i}$ , for  $i = 1, \dots, k$

$$\hat{q} = \Pi_{p,m}(q)$$

$$\hat{X}_h^0 = \sum_{i=p}^m f_m^i Z_i + f_m^{p-1} \hat{Z}_{p-1} \text{ et } \hat{Z}_{p-1} = \sum_{l=1}^k \phi_{p-1}^l \frac{\partial}{\partial x_{p-1}^l}$$

We denote by  $\hat{\Delta}_h$  the distribution generated by  $\hat{X}_h^0$  and  $X_m^1, \dots, X_m^k$ .

This comment will be used in a future paper about singular sets of special flags and their interpretations in terms of singularities kinematic evolution of an articulated arm.

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